

# HULLS AND HUSKS

JÁNOS KOLLÁR

Let  $X$  be a normal scheme and  $F$  a coherent sheaf on  $X$ . The *reflexive hull* or *double dual* of  $F$  is the sheaf  $F^{**} := \text{Hom}_X(\text{Hom}_X(F, \mathcal{O}_X), \mathcal{O}_X)$ . The natural map  $F \rightarrow F^{**}$  kills the torsion subsheaf of  $F$  and the support of  $\text{coker}[F \rightarrow F^{**}]$  has codimension  $\geq 2$ . This establishes a functor from the category of coherent sheaves on  $X$  to the category of reflexive coherent sheaves on  $X$ .

The same construction works as long as  $X$  satisfies Serre's condition  $S_2$ , but otherwise the double dual is not  $S_2$ . One can, however, define a natural functor from the category of quasi coherent sheaves on  $X$  to the category of quasi coherent sheaves on  $X$  that are  $S_2$  as sheaves over their support (17). We denote it by  $F \rightarrow F^{[*]}$  and call it the *hull* of  $F$ .

The main question we address in this section is the behaviour of the hull in families. The motivating example is the following theorem which describes all possible base changes that create a flat sheaf out of a non-flat sheaf.

**Theorem 1** (Flattening decomposition theorem). [Mum66, Lecture 8] *Let  $f : X \rightarrow S$  be a projective morphism and  $F$  a coherent sheaf on  $X$ . Then  $S$  can be written as a disjoint union of locally closed subschemes  $S_i \rightarrow S$  such that for any  $g : T \rightarrow S$  the following are equivalent:*

- (1) *the pull back of  $F$  to  $X \times_S T$  is flat over  $T$ , and*
- (2)  *$g$  factors through the disjoint union  $\coprod S_i \rightarrow S$ .*

Here we study a similar question where instead of the flatness of  $F$  we aim to understand the flatness of the hulls of the fiber-wise restrictions  $(F|_{X_s})^{[*]}$ . Note that even in very nice situations, for instance when  $f : X \rightarrow S$  is smooth and the restrictions  $F|_{X_s}$  are all torsion free, the hulls  $(F|_{X_s})^{**}$  do not form a sheaf on  $X$ . Thus the flattening decomposition theorem does not apply directly.

The main result (21) is a close analog of the flattening decomposition theorem for projective morphisms. The next formulation is somewhat vague; see (21) for the precise version.

**Theorem 2.** *Let  $f : X \rightarrow S$  be a projective morphism and  $F$  a coherent sheaf on  $X$ . Then  $S$  can be written as a disjoint union of locally closed subschemes  $S_i \rightarrow S$  such that for any  $g : T \rightarrow S$  the following are equivalent:*

- (1) *The hulls  $\{F_t^{[*]} : t \in T\}$  form a flat sheaf on  $X \times_S T \rightarrow T$ .*
- (2) *The map  $g$  factors through the disjoint union  $\coprod S_i \rightarrow S$ .*

The cases when the hull of each  $F_t^{\otimes m}$  is locally free of rank 1 for some  $m > 0$  has been treated in [Hac04, AH09]. In moduli theory, the main application of (2) is to the hulls  $\omega_X^{[m]}$  of  $\omega_X^{\otimes m}$ , see (24, 25). As a consequence we obtain a well defined theory of those deformations where the hulls  $\omega_X^{[m]}$  form a flat family.

The first step in the proof is the construction of the moduli space of *husks* and *quotient husks*. Generalizing the notion of a hull, a husk of a quasi coherent sheaf  $F$

is a map  $q : F \rightarrow G$  where  $G$  is torsion free over its support and  $q$  is an isomorphism at the generic points. There are no instability problems for coherent husks, and we prove that they have a fine moduli space (10). The necessary techniques are taken from [Gro62, LP93, LP95] with very little change. Similar ideas have been used in [Hon04, AK06, PT07, Ryd08].

It is then not difficult to identify the hulls among the husks to obtain (21).

The appendix discusses how to extend these results from projective morphisms to proper morphisms of algebraic spaces. This is joint work with M. Lieblich.

It would also be of interest to obtain a local version of the flattening decomposition theorem, but our methods are very much global in nature.

### Husks.

**Definition 3.** We say that a quasi coherent sheaf  $F$  on a scheme  $X$  is *pure* or *torsion free* over its support if every associated prime of  $F$  has dimension  $= \dim \text{Supp } F$ , that is, the maximum of the dimensions of the supports of local sections of  $F$ . In particular,  $\text{Supp } F$  is pure dimensional. We also say that  $F$  is pure of dimension  $n := \dim \text{Supp } F$ . If  $j : X \rightarrow Y$  is finite and  $F$  is pure then  $j_* F$  is also pure.

For a quasi coherent sheaf  $G$  on a scheme  $X$ , let  $\text{tors } G \subset G$  denote the subsheaf of those local sections whose support has dimension  $< \dim \text{Supp } G$ . Thus  $G/\text{tors } G$  is pure.

Let  $f : X \rightarrow S$  be a morphism and  $F$  a quasi coherent sheaf on  $X$ . We say that  $F$  is *pure* over  $S$  or that it has *torsion free fibers* over their support if for every  $s \in S$ , the restriction  $F_s$  is pure of the same dimension.

**Definition 4.** Let  $X$  be a scheme over a field  $k$ ,  $F$  a quasi coherent sheaf on  $X$  and  $n := \dim \text{Supp } F$ . A *husk* of  $F$  is a quasi coherent sheaf  $G$  together with a homomorphism  $q : F \rightarrow G$  such that

- (1)  $G$  is pure of dimension  $n$ , and
- (2)  $q : F \rightarrow G$  is an isomorphism at all  $n$ -dimensional points of  $X$ .

If  $h \in \text{Ann}(F)$  then  $h \cdot F = 0$ , hence  $h \cdot G \subset G$  is supported in dimension  $< n$ , hence 0. Thus  $G$  is also an  $\mathcal{O}_X/\text{Ann}(F)$  sheaf and so the particular choice of  $X$  matters very little.

Assume that  $X$  is projective and  $H$  is ample on  $X$ . As in (26), for a coherent sheaf  $M$  on  $X$  write

$$\chi(X, M(tH)) =: \sum a_i(M)t^i. \quad (4.3)$$

Set  $n := \dim \text{Supp } F$  and let  $G$  be a husk of  $F$ . If  $F, G$  are coherent then, by (26.1)

$$a_n(G) = a_n(F/\text{tors } F) \quad \text{and} \quad a_{n-1}(G) \geq a_{n-1}(F/\text{tors } F). \quad (4.4)$$

**5 (Universal husk).** The smallest husk of  $F$  is  $F/\text{tors } F$ .

There is also a largest or *universal husk*  $U(F)$  which can be constructed as follows. Let  $R$  be the total ring of quotients of  $\mathcal{O}_X/\text{Ann}(F)$ . That is, we invert every element that is a unit at every  $n$ -dimensional generic point of  $\mathcal{O}_X/\text{Ann}(F)$ . Then  $F \rightarrow F \otimes_X R$  is the universal husk.

First,  $F \otimes_X R$  is an  $R$ -sheaf, hence its associated primes are the  $n$ -dimensional generic points of  $\mathcal{O}_X/\text{Ann}(F)$ . By construction,  $F \rightarrow F \otimes_X R$  is an isomorphism at every  $n$ -dimensional generic point of  $\text{Supp } F$ .

Second, let  $F \rightarrow G$  be any other husk. Then we get  $F \otimes_X R \rightarrow G \otimes_X R$  which is an isomorphism at every  $n$ -dimensional generic point of  $\text{Supp } F$  hence an isomorphism

of  $R$ -sheaves. Thus  $F \rightarrow F \otimes_X R$  factors as  $F \rightarrow G \rightarrow F \otimes_X R$ . Since  $G$  has no lower dimensional associated primes,  $G \rightarrow F \otimes_X R$  is an injection. Hence the husks of  $F$  are the quasi coherent subsheaves of  $U(F)$  that contain  $F/\text{tors } F$ .

If  $n \geq 1$  then  $U(F)$  is never coherent, but it is the union of coherent husks. Thus a coherent sheaf has many different coherent husks and there is no universal coherent husk.

**Definition 6.** Let  $f : X \rightarrow S$  be a morphism and  $F$  a quasi coherent sheaf on  $X$ . Let  $n$  be the relative dimension of  $\text{Supp } F \rightarrow S$ . A *husk* of  $F$  is a quasi coherent sheaf  $G$  together with a homomorphism  $q : F \rightarrow G$  such that

- (1)  $G$  is flat and pure over  $S$ ,
- (2)  $q : F \rightarrow G$  is an isomorphism at every  $n$ -dimensional point of  $X_s \cap \text{Supp } F$  for every  $s \in S$ . Equivalently, if  $q_s : F_s \rightarrow G_s$  is a husk for every  $s \in S$ .

Note that the notion of a husk does depend on  $f$ .

As before,  $G$  is also an  $\mathcal{O}_X/\text{Ann}(F)$  sheaf and so  $X$  matters very little.

Husks are preserved by base change. That is, if  $g : T \rightarrow S$  is a morphism,  $X_T := X \times_S T$  and  $g_X : X_T \rightarrow X$  the first projection then  $g_X^* q : g_X^* F \rightarrow g_X^* G$  is also a husk.

**Lemma 7.** Let  $f : X \rightarrow S$  be a morphism and  $F$  a quasi coherent sheaf on  $X$ . Let  $q : F \rightarrow G$  be a husk of  $F$ .

- (1) Let  $g : X \rightarrow Z$  be a finite  $S$ -morphism. Then  $g_* G$  is a husk of  $g_* F$ .
- (2) Let  $h : Y \rightarrow X$  be a flat morphism. Then  $h^* G$  is a husk of  $h^* F$ .

*Proof.* If  $g$  is a finite morphism and  $M$  is a sheaf then the associated primes of  $g_* M$  are the images of the associated primes of  $M$ . This implies (1). Similarly, if  $h$  is flat then the associated primes of  $h^* M$  are the preimages of the associated primes of  $M$ , implying (2).  $\square$

**Definition 8.** Let  $f : X \rightarrow S$  be a morphism, and  $F$  a coherent sheaf on  $X$ . Let  $Husk(F)(*)$  be the functor that to a scheme  $g : T \rightarrow S$  associates the set of all coherent husks of  $g_X^* F$  with proper support over  $T$ , where  $g_X : T \times_S X \rightarrow X$  is the projection.

Let  $f : X \rightarrow S$  be a projective morphism,  $H$  an  $f$ -ample divisor and  $p(t)$  a polynomial. Let  $Husk_p(F)(*)$  be the functor that to a scheme  $g : T \rightarrow S$  associates the set of all coherent husks of  $g_X^* F$  with Hilbert polynomial  $p(t)$ .

**Definition 9.** Let  $f : X \rightarrow S$  be a morphism and  $F$  a quasi coherent sheaf on  $X$ . The husk of a quotient of  $F$  is called a *quotient husk* of  $F$ . Equivalently, a quotient husk of  $F$  is a quasi coherent sheaf  $G$  together with a homomorphism  $q : F \rightarrow G$  such that

- (1)  $G$  is pure over  $S$ , say of relative dimension  $m$  and
- (2)  $q : F \rightarrow G$  is surjective at all  $m$ -dimensional points of  $X_s \cap \text{Supp } G$  for every  $s \in S$ .

As in (8),  $QHusk_p(F)(*)$  denotes the functor that to a scheme  $g : T \rightarrow S$  associates the set of all coherent quotient husks of  $g_X^* F$  with Hilbert polynomial  $p(t)$ , where  $g_X : T \times_S X \rightarrow X$  is the projection.

The first existence theorem is the following.

**Theorem 10.** *Let  $f : X \rightarrow S$  be a projective morphism,  $H$  an  $f$ -ample divisor,  $p(t)$  a polynomial and  $F$  a coherent sheaf on  $X$ . Then  $\mathrm{QHusk}_p(F)$  is bounded, proper, separated and it has a fine moduli space  $\mathrm{QHusk}_p(F)$ .*

(The construction establishes  $\mathrm{QHusk}_p(F)$  as an algebraic space. There does not seem to be any obvious ample line bundle on it, so its projectivity may be a subtle question.)

Proof. We start by establishing the valuative criterion of properness and separatedness. Then we check that  $\mathrm{QHusk}_p(F)$  is bounded. The moduli space  $\mathrm{QHusk}_p(F)$  is then constructed using the theory of quotients by algebraic group actions.

As a preliminary step, note that the problem is local, thus we may assume that  $S$  is affine. Then  $f, X, F$  are defined over a finitely generated subalgebra of  $\mathcal{O}_S$ , hence we may assume in the sequel that  $S$  is of finite type.

#### 10.1 The valuative criterion of separatedness and properness.

Let  $T$  be the spectrum of a DVR with closed point  $0 \in T$  and generic point  $t \in T$ . Given  $g : T \rightarrow S$  we have  $g_X^* F$  where  $g_X : T \times_S X \rightarrow X$  is the projection as in (11). By assumption, we also have a husk  $q_t : F_t \rightarrow G_t$ ; set  $F'_t := \mathrm{im} q_t$ . By (30), there is a unique flat quotient  $g_X^* F \rightarrow F'$  that agrees with  $F'_t$  over the generic point.

Further, there is a closed subset  $B_t \subset \mathrm{Supp} G_t$  such that  $\dim B_t < \dim \mathrm{Supp} G_t$  and  $F'_t \rightarrow G_t$  is an isomorphism outside  $B_t$ . Let  $B_T \subset X_T$  be the closure of  $B_t$ .

$F'$  is flat over  $T$ , its generic fiber is pure and its special fiber is pure outside a subset  $Z_0 \subset X_0$  of dimension  $< \dim \mathrm{Supp} G_0$ . Furthermore,  $G_t$  and  $F'$  are naturally isomorphic over  $X_t \setminus B_t$ . Thus we can glue them to get a single sheaf  $G'$  defined on  $X_T \setminus (Z_0 \cup B_0)$ . By construction,  $G'$  is flat and pure over  $T$ .

Let  $j : X_T \setminus (Z_0 \cup B_0) \hookrightarrow X_T$  be the injection. Set  $G := j_* G'$ . Since  $Z_0 \cup B_0$  has codimension  $\geq 2$  in  $\mathrm{Supp} G_T$ , the push forward  $G$  is coherent by (35). The fibers of  $G$  are pure by (36.4). In particular,  $G$  is flat over  $T$  and  $G_0$  is a husk of  $F_0$ . By (36.4),  $G$  is the only extension of  $G'$  that is pure over  $T$ . These show that  $\mathrm{QHusk}(F)$  satisfies the valuative criterion of separatedness and properness.

Furthermore,  $G_0$  has the same Hilbert polynomial as  $G_t$ .

#### 10.2 Boundedness.

$F$  is  $m(F)$ -regular (31) for some  $m(F)$ . We show that all quotient husks  $q : F \rightarrow G$  are  $m$ -regular for some  $m$  depending only on  $m(F)$  and  $p(t)$ . This is a problem on the individual fibers, so from now on we assume that  $X \subset \mathbb{P}^n$  is a projective scheme over a field. We also use that this assertion holds for all quotients of  $F$  (30).

The proof is by induction on  $\deg p(t) = \dim \mathrm{Supp} G$ .

If  $\dim \mathrm{Supp} G = 0$  then any  $m$  works.

Assume next that  $\dim \mathrm{Supp} G = 1$ . Let  $G' \subset G$  be the image of  $F$ . The Hilbert polynomial of  $G'$  is  $p(t) - c$  where  $c$  is the length of  $G/G'$ . Pick any  $K'' \subset K := \ker q$  such that  $K/K''$  has length  $c$ . Then  $G'' := F/K''$  is a quotient of  $F$  with Hilbert polynomial  $p(t)$ , hence  $m$ -regular for some  $m$  depending only on  $m(F)$  and  $p(t)$ . Since  $G'' \rightarrow G$  has 0-dimensional kernel and cokernel, this implies that  $G$  is also  $m$ -regular.

Assume now that  $\dim \mathrm{Supp} G = n \geq 2$ . After a field extension, we may assume that the base field is infinite. Let  $H$  be a general hyperplane. Then  $F|_H$  is also  $m$ -regular and  $F|_H \rightarrow G|_H$  is a quotient husk with Hilbert polynomial  $p(t) - p(t-1)$

by (13). Hence, by induction,  $G|_H$  is  $m_1$ -regular for some  $m_1$  depending on  $m(F)$  and  $p(t)$ .  $G$  is then  $m$ -regular for some  $m$  depending on  $m(F)$  and  $p(t)$  by (32).

### 10.3 Construction of $\text{QHusk}_p(F)$ .

The existence of  $\text{QHusk}_p(F)$  is a local problem on  $S$ . As we noted in (6), we can replace  $X$  with any larger scheme. Thus we may assume that  $X = \mathbb{P}_S^n$ .

By boundedness, we can choose  $m$  such that for any quotient husk  $F \rightarrow G$  with Hilbert polynomial  $p(t)$ ,  $G_s(m)$  is generated by global sections and its higher cohomologies vanish. Thus each  $G_s(m)$  can be written as a quotient of  $\mathcal{O}_{X_s}^{\oplus p(m)}$ .

As in (30), let

$$Q_{p(t)} := \text{Quot}_{p(t)}^0(\mathcal{O}_X^{\oplus p(m)}) \subset \text{Quot}(\mathcal{O}_X^{\oplus p(m)})$$

be the universal family of quotients  $q_s : \mathcal{O}_{X_s}^{\oplus p(m)} \rightarrow M_s$  that have Hilbert polynomial  $p(t)$ , are pure, have no higher cohomologies and the induced map

$$q_s : H^0(X_s, \mathcal{O}_{X_s}^{\oplus p(m)}) \rightarrow H^0(X_s, M_s)$$

is an isomorphism.

Let  $\pi : Q_{p(t)} \rightarrow S$  be the structure map,  $\pi_X : Q_{p(t)} \times_S X \rightarrow X$  the second projection and  $M$  the universal sheaf on  $Q_{p(t)} \times_S X$ .

By (33) there is an open subscheme  $W_{p(t)} = \underline{\text{Hom}}^0(\pi_X^* F, M) \subset \underline{\text{Hom}}(\pi_X^* F, M)$  parametrizing those maps from  $\pi_X^* F$  to  $M$  that are surjective outside a subset of dimension  $\leq n-1$ . Let  $\sigma : W_{p(t)} \rightarrow Q_{p(t)}$  be the structure map, and  $\sigma_X : W_{p(t)} \times_S X \rightarrow Q_{p(t)} \times_S X$  the fiber product.

Note that  $W_{p(t)}$  parametrizes triples

$$w := \left[ F_w \xrightarrow{r_w} G_w \xleftarrow{q_w} \mathcal{O}_{X_w}(-m)^{\oplus p(m)} \right]$$

where  $r_w : F_w \rightarrow G_w$  is a quotient husk with Hilbert polynomial  $p(t)$  and  $q_w(m) : \mathcal{O}_{X_w}^{\oplus p(m)} \rightarrow G_w(m)$  is a surjection that induces an isomorphism on the spaces of global sections.

Let  $w' \in W_{p(t)}$  be another point corresponding to the triple

$$w' := \left[ F_{w'} \xrightarrow{r_{w'}} G_{w'} \xleftarrow{q_{w'}} \mathcal{O}_{X_{w'}}(-m)^{\oplus p(m)} \right].$$

such that

$$[F_w \xrightarrow{r_w} G_w] \cong [F_{w'} \xrightarrow{r_{w'}} G_{w'}].$$

Then the difference between  $w$  and  $w'$  comes from the different ways that we can write  $G_w \cong G_{w'}$  as quotients of  $\mathcal{O}_{X_w}(-m)^{\oplus p(t)}$ . Since we assume that  $q_w(m), q_{w'}(m)$  induce isomorphisms on the spaces of global sections, the different choices of  $q_w$  and  $q_{w'}$  correspond to different bases in  $H^0(X_w, G_w(m))$ . Thus the fiber of

$$\text{Mor}(*, W_{p(t)}) \rightarrow \text{QHusk}_p(F)(*) \quad \text{over} \quad \pi \circ \sigma(w) = \pi \circ \sigma(w') =: s \in S$$

is a principal homogeneous space under the group scheme

$$GL(p(t), k(s)) = \text{Aut}\left(H^0(X_s, G_s(m))\right).$$

Let  $G$  be the group scheme  $GL(p(t), S)$ . Then  $G$  acts on  $W_{p(t)}$  and  $\text{QHusk}_p(F) = W_{p(t)}/G$  [Kol97, KM97].  $\square$

**Definition 11.** Let  $f : X \rightarrow S$  be a morphism and  $F$  a quasi coherent sheaf on  $X$ . Let  $n = \max_{s \in S} \dim \text{Supp}(F|_{X_s})$ . We say that  $F$  is *generically flat* on every fiber of  $\text{Supp } F \rightarrow S$  if  $F$  is flat at every  $n$ -dimensional point of every fiber  $X_s$ . If  $F$  is coherent, then this is equivalent to the following:

There is a subscheme  $Z \subset X$  such that

- (1)  $F|_{X \setminus Z}$  is flat over  $S$ , and
- (2)  $\dim(X_s \cap Z) < n$  for every  $s \in S$ .

**Corollary 12.** *Notation and assumptions as in (10).*

- (1)  $\text{Husk}_p(F)$  is bounded, separated and it has a fine moduli space  $\text{Husk}_p(F)$  which is an open subspace of  $\text{QHusk}_p(F)$ .
- (2) Assume that  $F$  is generically flat on every fiber of  $\text{Supp } F \rightarrow S$ . Then  $\text{Husk}_p(F)$  is proper and  $\text{Husk}_p(F) \subset \text{QHusk}_p(F)$  is closed.

Proof. Let  $q_U : \pi^* F \rightarrow G_U$  be the universal quotient husk over  $\pi : \text{QHusk}_p(F) \rightarrow S$ . For a surjective map with flat target, it is an open condition to be fiber-wise isomorphic (cf. [Mat86, 22.5]). Thus there is a closed subset  $Z \subset \text{QHusk}_p(F) \times_S X$  such that  $q_U : \pi^* F \rightarrow G_U$  is a fiberwise isomorphism exactly outside  $Z$ . Then  $\text{Husk}_p(F) \subset \text{QHusk}_p(F)$  is the largest open subset over which the fiber dimension of  $Z \rightarrow \text{QHusk}_p(F)$  is less than  $\deg p(t)$ .

Assume next that  $F$  is generically flat on every fiber of  $\text{Supp } F \rightarrow S$ . Then there is a closed subscheme  $W \subset \text{QHusk}_p(F) \times_S X$  such that the fiber dimension of  $W \rightarrow \text{QHusk}_p(F)$  is less than  $\deg p(t)$ , and  $F$  is flat and  $q$  is surjective outside  $W$ . Then  $\ker q$  is also flat outside  $W$ , hence  $\dim \text{Supp } \ker q_s < n$  is a closed condition. Note that  $q_s : F_s \rightarrow G_s$  is a husk iff  $q_s$  is generically injective, that is iff  $\dim \text{Supp } \ker q_s < n$ . This proves (2).  $\square$

**13** (Restriction of husks). Let  $q : F \rightarrow G$  be a husk or a quotient husk and  $H \subset X$  a Cartier divisor. When is  $F|_H \rightarrow G|_H$  a husk or a quotient husk?

First of all, we need that  $G|_H$  be flat and pure of dimension  $(n-1)$ . The first of these holds if  $H$  does not contain any associated prime of any  $G_s$ , cf. [Mat86, Thm.22.5]. Since  $(G|_H)|_s = G_s|_H$ , the second condition is satisfied if  $H$  does not contain any associated prime of any of the hulls  $G_s^{[**]}$ ; see (34) and (14).

Second, for any  $s \in S$ , we have an exact sequence

$$0 \rightarrow K_s \rightarrow F_s \rightarrow G_s \rightarrow G_s/F_s \rightarrow 0.$$

Tensoring with  $\mathcal{O}_H$  is exact if  $\text{Tor}^1(F_s/K_s, \mathcal{O}_H) = 0$  and  $\text{Tor}^1(G_s/F_s, \mathcal{O}_H) = 0$ . Every associated prime of  $F_s/K_s$  is an associated prime of  $G_s$ . Thus both vanishings hold if  $H$  does not contain any associated prime of  $G_s$  or of  $G_s/F_s$  for every  $s \in S$ .

In particular, if  $G$  is coherent and the residue fields of  $S$  are infinite, then these conditions hold for general members of any base point free linear system on  $X$ . (See (16) for the required coherence of  $G_s^{[**]}$ .)

## Hulls.

**Definition 14.** Let  $X$  be a scheme over a field  $k$  and  $F$  a quasi coherent sheaf on  $X$ . Set  $n := \dim \text{Supp } F$ . A husk  $q : F \rightarrow G$  is called *tight* if  $q$  is onto at all  $(n-1)$ -dimensional points of  $X$ . There is a unique maximal tight husk  $q : F \rightarrow F^{[**]}$ , called the *hull* (or  $S_2$ -hull, see (15)) of  $F$ .

$F^{[**]}$ , as a subset of the universal husk  $U(F)$  defined in (5), is generated by all local sections  $\phi \in U(F)$  such that  $\phi$  is a local section of  $q(F)$  at all  $(n-1)$ -dimensional points of  $\text{Supp } G$ .

If  $X$  itself is normal,  $F$  is coherent and  $\text{Supp } F = X$ , then  $F^{[**]}$  is the usual reflexive hull  $F^{**}$  of  $F$ .

The hull of a nonzero sheaf is also nonzero, in contrast with the reflexive hull which kills all torsion sheaves.

**Lemma 15.** *Let  $X$  be a scheme over a field  $k$  and  $F$  a coherent sheaf on  $X$ .*

- (1) *Let  $r : F \rightarrow G$  be any tight husk. Then  $q : F \rightarrow F^{[**]}$  extends uniquely to an injection  $q_G : G \hookrightarrow F^{[**]}$ .*
- (2)  *$F^{[**]}$  is the unique tight husk which is  $S_2$  over its support.*
- (3)  *$F^{[**]}$  is the smallest  $S_2$  husk of  $F$ . That is, if  $r : F \rightarrow G$  is any husk such that  $G$  is  $S_2$  over its support, then  $r$  factors as  $F \rightarrow F^{[**]} \hookrightarrow G$ .*
- (4) *Let  $Z \subset X$  be a closed subset such that  $\dim Z \leq n-2$  and  $F/\text{tors } F$  is  $S_2$  over  $X \setminus Z$ . Let  $j : X \setminus Z \rightarrow X$  denote the injection. Then*

$$F^{[**]} = j_*((F/\text{tors } F)|_{X \setminus Z}).$$

- (5) *Assume that  $X$  is projective,  $H$  is ample on  $X$ ,  $F \rightarrow G$  is any coherent husk and  $n = \dim \text{Supp } F$ . Then,*
  - (a)  $a_n(F^{[**]}) = a_n(F/\text{tors } F)$  and  $a_{n-1}(F^{[**]}) = a_{n-1}(F/\text{tors } F)$ .
  - (b)  $a_n(F^{[**]}) = a_n(G)$  and  $a_{n-1}(F^{[**]}) \leq a_{n-1}(G)$ ,
  - (c) *equality holds iff  $G \subset F^{[**]}$ .*

*Thus the hull minimizes  $a_{n-1}$  and maximizes the rest of the Hilbert polynomial.*

*Proof.* The first property holds by definition.

Let  $r : F \rightarrow G$  be a tight husk such that  $G \subsetneq F^{[**]}$ . Pick any  $\phi \in F^{[**]} \setminus G$  and a function  $f \in \mathcal{O}_X$  which is invertible at all  $n$ -dimensional generic points of  $\text{Supp } F$  such that  $f\phi \in G$ . Then  $f\phi \in G/fG$  has  $\leq (n-2)$ -dimensional support, thus  $G$  is not  $S_2$ .

Conversely, with  $f$  as above, let  $\phi \in F^{[**]}/fF^{[**]}$  be a local section which has  $\leq (n-2)$ -dimensional support. Then  $\langle F^{[**]}, f^{-1}\phi \rangle \in U(F)$  is also a tight husk of  $F$ . Thus  $\phi \in fF^{[**]}$  and so  $F^{[**]}/fF^{[**]}$  has no nonzero local sections with  $\leq (n-2)$ -dimensional support. Thus  $F^{[**]}$  is  $S_2$ , hence (2) holds.

Let  $r : F \rightarrow G \subset U(F)$  be a husk which is  $S_2$ . Pick any local section  $\phi \in F^{[**]}$ . Then  $\langle G, \phi \rangle/G$  is supported in dimension  $\leq (n-2)$ . Since  $G$  is  $S_2$ , this implies that  $\phi \in G$ , proving (3).

(4) is discussed in greater detail in (36).

(5.a) follows from (26.1) and, together with (4.4), it implies (5.b). If  $a_n(F^{[**]}) = a_n(G)$  and  $a_{n-1}(F^{[**]}) = a_{n-1}(G)$ , then, by (26.2),  $F \rightarrow G$  is a tight husk, hence  $G \subset F^{[**]}$  by (1).  $\square$

**16** (Hulls of coherent sheaves). The hull  $F^{[**]}$  of a coherent sheaf  $F$  is almost always coherent. For instance, this holds if  $X$  is of finite type over a field or over an excellent ring.

To see this, we can assume that  $X$  is affine and replace  $F$  by  $F/\text{tors } F$ . Then there is a sequence of subsheaves  $0 = F_0 \subset \cdots \subset F_n = F$  such that every  $F_{m+1}/F_m$  is isomorphic to an ideal sheaf in  $\mathcal{O}_{X_m}$  for some integral subscheme  $X_m \subset X$  of dimension  $n$ .

By (15.4),  $F \rightarrow F^{[**]}$  is left exact on sequences of pure  $n$ -dimensional sheaves. Thus it is sufficient to prove that the hull of any ideal sheaf is coherent. In turn this follows if the hull of  $\mathcal{O}_{X_m}$  is coherent. By (15.3) the hull of  $\mathcal{O}_{X_m}$  is contained in the normalization of  $\mathcal{O}_{X_m}$ . Thus  $\mathcal{O}_{X_m}^{[**]}$  is coherent whenever the normalization is coherent.

**Definition 17.** Let  $f : X \rightarrow S$  be a morphism and  $F$  a quasi coherent sheaf. Let  $n$  be the relative dimension of  $\text{Supp } F \rightarrow S$ . A *hull* of  $F$  is a husk  $q : F \rightarrow G$  such that, for every  $s \in S$ , the induced map  $q_s : F_s \rightarrow G_s$  is a hull (14).

We see in (18) that a hull is unique if it exists. Note that if a hull exists then  $F$  is generically flat on every fiber of  $\text{Supp } F \rightarrow S$ .

It is clear from the definition that hulls are preserved by base change. That is, if  $g : T \rightarrow S$  is a morphism,  $X_T := X \times_S T$  and  $g_X : X_T \rightarrow X$  the first projection then  $g_X^* q : g_X^* F \rightarrow g_X^* G$  is also a hull.

**Lemma 18.** *Let  $f : X \rightarrow S$  be a morphism of finite type and  $F$  a coherent sheaf on  $X$ . Let  $n$  be the relative dimension of  $\text{Supp } F \rightarrow S$ .*

- (1) *Let  $q : F \rightarrow G$  be a hull and set  $Z := \text{Supp } G/F$ . Then  $G$  is coherent,  $\dim(X_s \cap Z) \leq n - 2$  for every  $s \in S$  and  $F/\text{tors } F$  is flat over  $X \setminus Z$ .*
- (2) *Conversely, let  $Z \subset X$  be any closed subset such that  $\dim(X_s \cap Z) \leq n - 2$  for every  $s \in S$  and  $F/\text{tors } F$  is flat over  $X \setminus Z$ . Let  $j : X \setminus Z \rightarrow X$  denote the injection. Then*

$$G = j_*((F/\text{tors } F)|_{X \setminus Z}).$$

*In particular,  $F$  has at most one hull.*

Proof.  $G_s = (F|_s)^{[**]}$  is coherent by (16), thus  $G$  is coherent by the Nakayama lemma. The rest of the first part is clear from the definition. To see the converse, let  $q : F \rightarrow G$  be a hull and  $Z$  any closed subset such that  $\dim(X_s \cap Z) \leq n - 2$  for every  $s \in S$  and  $Z \supset \text{Supp } \text{coker } q$ . Then  $F/\text{tors } F$  and  $G$  are isomorphic over  $X \setminus Z$ , hence  $F/\text{tors } F$  is flat over  $X \setminus Z$ . Furthermore, by (36),

$$G = j_*(G|_{X \setminus Z}) = j_*((F/\text{tors } F)|_{X \setminus Z}).$$

Thus  $G$  is unique. □

**Definition 19.** Let  $f : X \rightarrow S$  be a projective morphism and  $F$  a coherent sheaf on  $X$ . For a scheme  $g : T \rightarrow S$  set  $\text{Hull}(F)(T) = 1$  if  $g_X^* F$  has a hull and  $\text{Hull}(F)(T) = \emptyset$  if  $g_X^* F$  does not have a hull, where  $g_X : T \times_S X \rightarrow X$  is the projection.

**Definition 20.** A morphism  $g : \bar{S} \rightarrow S$  is a *locally closed decomposition* of  $S$  if

- (1) for every connected component  $\bar{S}_i \subset \bar{S}$ , the restriction of  $g$  to  $\bar{S}_i$  is a locally closed embedding, and
- (2)  $g$  is one-to-one and onto on geometric points.

The second existence theorem is the following.

**Theorem 21** (Flattening decomposition for hulls). *Let  $f : X \rightarrow S$  be a projective morphism and  $F$  a coherent sheaf on  $X$ . Then*

- (1)  *$\text{Hull}(F)$  is bounded, separated and it has a fine moduli space  $\text{Hull}(F)$ .*
- (2) *The structure map  $\text{Hull}(F) \rightarrow S$  is a locally closed decomposition.*



Proof. We construct the locally closed decomposition  $\text{Hull}(F) \rightarrow S$  by first identifying a closed stratum and then using induction.

Let  $n$  be the maximal fiber dimension of  $\text{Supp } F \rightarrow S$ .

For any point  $s \in S$  write

$$\chi(X_s, (F_s)^{[*]}(tH)) =: p_s(t) =: a_n(s)t^n + a_{n-1}(s)t^{n-1} + O(t^{n-2}).$$

By (1) and (28.3),  $\{p_s(t) : s \in S\}$  is a finite set of polynomials. Let  $p(t) = a_nt^n + a_{n-1}t^{n-1} + O(t^{n-2})$  be the polynomial which lexicographically maximizes the triple  $(a_n(s), -a_{n-1}(s), p_s)$  for all  $s \in S$ . (Note the minus sign before  $a_{n-1}(s)$ .)

(21.3) *Claim.* Every quotient husk of  $F$  with Hilbert polynomial  $p(t)$  is a hull.

Proof. This holds after any base change, but, for simplicity of notation, we work directly over  $S$ .

Let  $F \rightarrow G$  be a quotient husk with Hilbert polynomial  $p(t)$ . The following exact sequences define  $K$  and  $F'$ :

$$0 \rightarrow F' \rightarrow G \rightarrow G/F' \rightarrow 0 \quad \text{and} \quad 0 \rightarrow K \rightarrow F \rightarrow F' \rightarrow 0.$$

Since  $G$  is flat over  $S$  and the fiber dimension of  $\text{Supp}(G/F') \rightarrow S$  is less than  $n$ , see that  $F'$  is flat over  $S$  at the generic points of its support in each fiber.

Therefore, for every  $s \in S$ ,  $K_s \rightarrow F_s \rightarrow F'_s \rightarrow 0$  is exact at all generic points of  $\text{Supp } G_s$ . Thus  $a_n(F_s) \geq a_n(F'_s)$ . On the other hand, we assumed that  $a_n(F'_s) = a_n(G_s)$  is the largest possible. Thus  $a_n(F_s) = a_n(F'_s)$  and so  $\text{Supp } K \rightarrow S$  has fiber dimension  $< n$  over  $S$ . In particular,  $F \rightarrow G$  is a husk.

Since  $F_s \rightarrow G_s$  is a husk,  $a_{n-1}(F_s) \leq a_{n-1}(G_s)$  by (15.5). By our choice,  $a_{n-1}(G_s)$  is the smallest possible, hence  $a_{n-1}(F_s) = a_{n-1}(G_s)$  and so  $G_s \subset F_s^{[*]}$  by (15.5). Since  $p(t)$  is maximized, this implies that  $G_s = F_s^{[*]}$ . Thus  $F \rightarrow G$  is a hull.  $\square$

By (10),  $\text{QHusk}_p(F) \rightarrow S$  is proper. As we proved, it parametrizes hulls, hence  $\text{QHusk}_p(F) \rightarrow S$  is a monomorphism (22, 18.2). A proper monomorphism is a closed embedding (22); let  $S_p \subset S$  denote the image of  $\text{QHusk}_p(F) \rightarrow S$ .

We can now replace  $S$  by  $S \setminus S_p$  and conclude by induction on the cardinality of  $\{p_s(t) : s \in S\}$ .  $\square$

**Definition 22** (Monomorphisms). A morphism of schemes  $f : X \rightarrow Y$  is a *monomorphism* if for every scheme  $Z$  the induced map of sets  $\text{Mor}(Z, X) \rightarrow \text{Mor}(Z, Y)$  is an injection.

By [Gro67, IV.17.2.6] this is equivalent to assuming that  $f$  be universally injective and unramified.

A closed or open embedding is a monomorphism. Other typical example of monomorphisms is the normalization of the node with a point missing, that is  $\mathbb{A}^1 \setminus \{-1\} \rightarrow (y^2 = x^3 + x^2)$  given by  $(t \mapsto (t^2 - 1, t^3 - t))$ .

A proper monomorphism  $f : Y \rightarrow X$  is a closed embedding. Indeed, a proper monomorphism is injective on geometric points, hence finite. Thus it is a closed embedding iff  $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$  is onto. By the Nakayama lemma this is equivalent to  $f_x : f^{-1}(x) \rightarrow x$  being an isomorphism for every  $x \in f(Y)$ . By passing to geometric points, we are down to the case when  $X = \text{Spec } k$ ,  $k$  is algebraically closed and  $Y = \text{Spec } A$  where  $A$  is an Artin  $k$ -algebra.

If  $A \neq k$  then there are at least 2 different  $k$  maps  $A \rightarrow k[\epsilon]$ , thus  $\text{Spec } A \rightarrow \text{Spec } k$  is not a monomorphism.

D. Rydh pointed out that, besides hulls, it is also of interest to consider  $F/\text{tors } F$ , which is the smallest husk of  $F$ . In this case, the method of (21) gives the following if we first maximize  $a_n$  and then minimize  $p(t)$ .

**Proposition 23** (Flattening decomposition for pure quotients). *Let  $f : X \rightarrow S$  be a projective morphism and  $F$  a coherent sheaf on  $X$ . Then  $S$  can be written as a disjoint union of locally closed subschemes  $S_i \rightarrow S$  such that for any  $g : T \rightarrow S$  the following are equivalent:*

- (1)  $g_X^* F / \text{tors } g_X^* F$  is flat and pure.
- (2)  $g$  factors through the disjoint union  $\coprod S_i \rightarrow S$ .

### Applications.

Applying (21) to the relative dualizing sheaf gives the following result.

**Corollary 24.** *Let  $f : X \rightarrow S$  be projective and equidimensional. Let  $Z \subset X$  be a closed subscheme such that  $\text{codim}(X_s, Z \cap X_s) \geq 2$  for every  $s \in S$  and  $(X \setminus Z) \rightarrow S$  is flat with Gorenstein fibers. Then, for any  $m$  there is a locally closed decomposition  $S_m \rightarrow S$  such that for any  $g : T \rightarrow S$  the following are equivalent*

- (1)  $\omega_{X \times_S T/T}^{[m]}$  is flat over  $T$  and commutes with base change.
- (2)  $g$  factors through  $S_m \rightarrow S$ .

Proof. The question is local on  $S$ , thus we may assume that there is a finite surjection  $\pi : X \rightarrow \mathbb{P}_S^n$ . One can now define  $\omega_{X/S}$  as

$$\omega_{X/S} := \text{Hom}_{\mathbb{P}_S^n}(\pi_* \mathcal{O}_X, \omega_{\mathbb{P}_S^n/S}).$$

In general,  $\omega_{X/S}$  does not commute with base change but, by assumption, its restriction to  $X \setminus Z$  is locally free.

We claim that  $S_m = \text{Hull}(\omega_{X/S}^{\otimes m})$ .

Given  $g : T \rightarrow S$ , let  $j_T : X \times_S T \setminus Z \times_S T \rightarrow X \times_S T$  be the inclusion. Then

$$\omega_{X \times_S T/T}^{[m]} = (j_T)_* g_X^* \omega_{X \setminus Z/S}^{\otimes m}.$$

If  $T \mapsto \omega_{X \times_S T/T}^{[m]}$  commutes with restrictions to the fibers of  $X \times_S T \rightarrow T$ , then  $\omega_{X \times_S T/T}^{[m]}$  has  $S_2$  fibers, hence  $\omega_{X \times_S T/T}^{[m]}$  is the hull of  $\omega_{X \times_S T/T}^{\otimes m}$ .

Conversely, by (18), if  $\omega_{X \times_S T/T}^{\otimes m}$  has a hull then it is  $\omega_{X \times_S T/T}^{[m]}$  and it commutes with further base changes by (17).  $\square$

**Corollary 25.** *Let  $f : X \rightarrow S$  be projective and equidimensional. Let  $Z \subset X$  be a closed subscheme such that  $\text{codim}(X_s, Z \cap X_s) \geq 2$  for every  $s \in S$  and  $(X \setminus Z) \rightarrow S$  is flat with Gorenstein fibers. Assume in addition that there is an  $N > 0$  such that  $\omega_{X_s}^{[N]}$  is locally free for every  $s \in S$ .*

*Then there is a locally closed decomposition  $S^* \rightarrow S$  such that a morphism  $g : T \rightarrow S$  factors through  $S^*$  iff  $\omega_{X \times_S T/T}^{[m]}$  is flat over  $T$  and commutes with base change for every  $m \in \mathbb{Z}$ .*

Proof. Let  $S_i \rightarrow S$  be as in (24). Take  $S^*$  to be the fiber product of the morphisms  $S_1 \rightarrow S, \dots, S_N \rightarrow S$ .  $\square$

### Hilbert polynomials of non-flat sheaves.

**26** (Hilbert polynomials). Let  $X$  be a projective scheme of dimension  $n$  and  $H$  an ample divisor. For a coherent sheaf  $F$ , write its Hilbert polynomial as

$$\chi(X, F(tH)) = a_n(F)t^n + \cdots + a_0(F).$$

Then  $a_n(F) \geq 0$  and  $a_n(F) = 0$  iff  $\dim \text{Supp } F < n$ .

(26.1) Let  $u : F \rightarrow G$  be a map of coherent sheaves which is an isomorphism outside a subset  $Z \subset X$  of dimension  $\leq n - r$ . Then  $a_i(F) = a_i(G)$  for  $n \geq i > n - r$  and if  $\dim \text{Supp } \ker u < n - r$  then  $a_{n-r}(G) \geq a_{n-r}(F)$ . Indeed note that

$$\chi(X, F(tH)) - \chi(X, G(tH)) = \chi(X, (\ker u)(tH)) - \chi(X, (\text{coker } u)(tH)).$$

By assumption, both  $\text{coker } u$  and  $\ker u$  are supported on  $Z$ , hence their Hilbert polynomials have degree  $\leq n - r$ .

(26.2) Conversely, let  $u : F \rightarrow G$  be a map of sheaves which is an isomorphism at the generic points. If  $a_i(F) = a_i(G)$  for  $n \geq i > n - r$  and  $\dim \text{Supp } \text{tors } F \leq n - r$ , then  $u$  is an isomorphism outside a subset of dimension  $\leq n - r$ .

(26.3) Let  $f : X \rightarrow S$  be a projective morphism of pure relative dimension  $n$ . Let  $F$  be a sheaf on  $X$ . Fix an integer  $r$  and assume that there is a closed subscheme  $Z \subset X$  such that  $F$  is flat over  $X \setminus Z$  and  $\dim_s(Z \cap X_s) \leq n - r$  for every  $s \in S$ .

Pick an  $f$ -very ample divisor  $H$ . Every  $s \in S$  has an open neighborhood  $U$  such that for general  $H_0, \dots, H_{n-r} \in |H_{f^{-1}(U)}|$ , the restriction  $F|_{H_0 \cap \cdots \cap H_{n-r}}$  is flat over  $U$ . In particular, the Hilbert polynomial  $\chi(H_0 \cap \cdots \cap H_{n-r}, F(m))$  is well defined.

For each  $u \in U$ , the Hilbert polynomial  $\chi(H_0 \cap \cdots \cap H_{n-r} \cap X_u, F(m))$  determines the top  $r$  coefficients of the Hilbert polynomials  $\chi(X_u, F(t)|_{X_u})$ . Thus we conclude the following.

(26.4) Under the assumptions of (26.3), the top  $r$  coefficients of the Hilbert polynomials  $\chi(X_s, F(t)|_{X_s})$  are locally constant on  $S$ .

**Definition 27.** Let  $p_1$  and  $p_2$  be two polynomials. We say that  $p_1 \leq p_2$  if  $p_1(t) \leq p_2(t)$  for all  $t \gg 0$ .

For example, if  $F_1 \subset F_2$  are coherent sheaves on a projective scheme and  $p_i$  is the Hilbert polynomial of  $F_i$  then  $p_1 \leq p_2$  and equality holds iff  $F_1 = F_2$ .

**Proposition 28.** Let  $f : X \rightarrow S$  be a projective morphism and  $H$  an  $f$ -ample Cartier divisor. Let  $F$  be a coherent sheaf on  $X$ . For a point  $s \in S$ , set  $F_s := F|_{X_s}$ .

(1) The Hilbert polynomial function

$$s \mapsto \chi(X_s, F_s(tH))$$

is constructible and upper semi continuous on  $S$ .

(2) Assume that  $F$  is generically flat on every fiber of  $\text{Supp } F \rightarrow S$ . Then the Hilbert polynomial

$$s \mapsto \chi(X_s, (F_s / \text{tors } F_s)(tH))$$

is constructible and lower semi continuous on  $S$ .

(3) If  $F$  is generically flat on every fiber of  $\text{Supp } F \rightarrow S$  and  $\dim \text{Supp } \text{tors } F_s \leq \dim \text{Supp } F_s - 2$  for every  $s \in S$  then the Hilbert polynomial of the hull

$$s \mapsto \chi(X_s, (F_s)^{[*]}(tH))$$

is constructible and upper semi continuous on  $S$ .

Proof. We may assume that  $S$  is reduced. By generic flatness [Mum66, Lec.8] there is a dense open subset  $S^0 \subset S$  such that  $F$  and  $F/\text{tors } F$  are flat over  $S^0$  and  $F/\text{tors } F$  has pure fibers. Thus  $\chi(X_s, F_s(tH))$  and  $\chi(X_s, (F_s/\text{tors } F_s)(tH))$  are both locally constant on  $S^0$ . By Noetherian induction we conclude that both functions are constructible.

It is thus enough to check semi continuity when  $S$  is the spectrum of a DVR. Let  $0 \in S$  be the closed point and  $g \in S$  the general point. Let  $\text{tors}_0 F \subset F$  be the torsion supported on  $X_0$ . Then  $F/\text{tors}_0 F$  is flat over  $S$  and so

$$\chi(X_g, F_g(tH)) = \chi(X_0, (F/\text{tors}_0 F) \otimes \mathcal{O}_{X_0}(tH)).$$

There is an exact sequence

$$(\text{tors}_0 F) \otimes \mathcal{O}_{X_0} \rightarrow F \otimes \mathcal{O}_{X_0} \rightarrow (F/\text{tors}_0 F) \otimes \mathcal{O}_{X_0} \rightarrow 0,$$

hence

$$\chi(X_0, F \otimes \mathcal{O}_{X_0}(tH)) \geq \chi(X_0, (F/\text{tors}_0 F) \otimes \mathcal{O}_{X_0}(tH)) = \chi(X_g, F_g(tH)),$$

which proves upper semi continuity in the first case.

If  $F$  is generically flat on every fiber of  $\text{Supp } F \rightarrow S$ , then

$$\dim \text{Supp}(\text{tors}_0 F) \otimes \mathcal{O}_{X_0} < \dim \text{Supp } F_0.$$

Thus  $(\text{tors}_0 F) \otimes \mathcal{O}_{X_0}$  maps to the torsion subsheaf of  $F_0$  and  $F_0/\text{tors } F_0$  is also a quotient of  $(F/\text{tors}_0 F) \otimes \mathcal{O}_{X_0}$ . Therefore

$$\chi(X_0, (F_0/\text{tors } F_0)(tH)) \leq \chi(X_0, (F/\text{tors}_0 F) \otimes \mathcal{O}_{X_0}(tH)) = \chi(X_g, F_g(tH)),$$

which proves lower semi continuity in the second case.

In order to prove (28.3), let  $j : U \hookrightarrow X$  be an open set such that  $F/\text{tors } F$  is torsion free and flat over  $U$  and  $(X \setminus U) \cap X_{s_g}$  has codimension  $\geq 2$  for a generic point  $s_g \in S$ . There is an open neighborhood  $U_1$  of  $s_g$  such that  $(X \setminus U) \cap X_s$  has codimension  $\geq 2$  for every point  $s \in U_1$ . Furthermore,  $j_*(F|_U)$  is flat and has  $S_2$  fibers over a nonempty open  $U_2 \subset U_1$ . Thus

$$\mathcal{O}_{X_s} \otimes j_*(F|_U) \cong (F|_{X_s})^{[**]},$$

and so  $\chi(X_s, (F_s)^{[**]}(tH))$  is locally constant on  $U_2$ . As before, this proves constructibility.

As before, it is enough to check upper semi continuity when  $S$  is the spectrum of a DVR and  $F$  is torsion free. In particular,  $F$  is flat over  $S$ .

Let  $j : U \hookrightarrow X$  be an open set such that  $F$  is flat over  $U$ ,  $(X \setminus U) \cap X_g$  has codimension  $\geq 2$  and  $(X \setminus U) \cap X_0$  has codimension  $\geq 1$ . Then  $G := j_*(F|_U)$  is flat over  $S$  and  $F \rightarrow G$  is generically an isomorphism on every fiber. On the generic fiber,  $G_g \cong (F_g)^{[**]}$ . On the special fiber we know that  $G_0$  is torsion free and there is a map  $F_0 \rightarrow G_0$  which is an isomorphism at all generic points.

Since  $F$  and  $G$  are both flat over  $S$ ,

$$\chi(X_0, F_0(tH)) = \chi(X_g, F_g(tH)) \quad \text{and} \quad \chi(X_0, G_0(tH)) = \chi(X_g, G_g(tH)).$$

Furthermore, since  $F_g$  is torsion free,

$$\deg_t \left( \chi(X_g, G_g(tH)) - \chi(X_g, F_g(tH)) \right) \leq n - 2,$$

hence also

$$\deg_t \left( \chi(X_0, G_0(tH)) - \chi(X_0, F_0(tH)) \right) \leq n - 2.$$

By assumption,  $\dim \text{Supp tors } F_0 \leq n - 2$ , hence  $F_0 \rightarrow G_0$  is an isomorphism at all codimension 1 points by (26.2). Hence by (15.1) there is an injection  $G_0 \hookrightarrow (F_0)^{[*]}$ . Thus

$$\chi(X_0, (F_0)^{[*]}(tH)) \geq \chi(X_0, G_0(tH)) = \chi(X_g, G_g(tH)) = \chi(X_g, (F_g)^{[*]}(tH)) \quad \square$$

**Example 29.** The condition on  $\dim \text{Supp tors } F_s$  in (28.3) is necessary. Let  $C, S$  be smooth projective curves and  $X = C \times S$ . Let  $f : X \rightarrow S$  be the projection and  $F$  the ideal sheaf of a point  $(c, s_0) \in X$ . For  $s \neq s_0$ ,  $F_s \cong \mathcal{O}_C$  has Hilbert polynomial  $t \deg H + 1 - g(C)$ . On the other hand,  $F_{s_0} \cong \mathcal{O}_C(-c) + k(c)$  and its hull is  $\mathcal{O}_C(-c)$  with Hilbert polynomial  $t \deg H - g(C)$ .

### Quot-schemes.

**30** (Quot-schemes). [Gro62] Let  $f : X \rightarrow S$  be a morphism and  $F$  a coherent sheaf on  $X$ .  $\text{Quot}(F)(*)$  denotes the functor that to a scheme  $g : T \rightarrow S$  associates the set of all quotients of  $g_X^* F$  that are flat over  $T$  with proper support, where  $g_X : T \times_S X \rightarrow X$  is the projection.

If  $F = \mathcal{O}_X$ , then a quotient can be identified with a subscheme of  $X$ , thus  $\text{Quot}(\mathcal{O}_X) = \text{Hilb}(X)$ , the Hilbert functor.

If  $H$  is an  $f$ -ample divisor and  $p(t)$  a polynomial, then  $\text{Quot}_p(F)(*)$  denotes those flat quotients that have Hilbert polynomial  $p(t)$ .

By [Gro62],  $\text{Quot}_p(F)$  is bounded, proper, separated and it has a fine moduli space  $\text{Quot}_p(F)$ . See [Ser06, Sec.4.4] for a detailed proof. If  $F = \mathcal{O}_X$ , then  $\text{Quot}(\mathcal{O}_X) = \text{Hilb}(X)$ , the Hilbert scheme of  $X$ .

Note that one can write  $F$  as a quotient of  $\mathcal{O}_{\mathbb{P}^n}(-m)^r$  for some  $m, r$ , thus  $\text{Quot}_p(F)$  can be viewed as a subscheme of  $\text{Quot}(\mathcal{O}_{\mathbb{P}^n}^r)$ . The theory of  $\text{Quot}(\mathcal{O}_{\mathbb{P}^n}^r)$  is essentially the same as the study of the Hilbert functor, discussed in [Mum66] and [Kol96, Sec.I.1].

**31** (Castelnuovo-Mumford regularity). Let  $F$  be a coherent sheaf on  $\mathbb{P}^n$ . We say that  $F$  is  $m$ -regular if  $H^i(\mathbb{P}^n, F(m-i)) = 0$  for  $i \geq 1$ . See [Laz04, Sec.I.1.8] for a detailed treatment.

It is known that if  $F$  is  $m$ -regular then it is also  $m'$ -regular for every  $m' \geq m$  and the multiplication maps

$$H^0(\mathbb{P}^n, F(m')) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \rightarrow H^0(\mathbb{P}^n, F(m'+1))$$

are surjective for  $m' \geq m$ . Thus  $F(m)$  is generated by global sections and so  $F$  is a quotient of the sum of  $h^0(\mathbb{P}^n, F(m))$  copies of  $\mathcal{O}_{\mathbb{P}^n}(-m)$ . In particular, all  $m$ -regular sheaves with Hilbert polynomial  $p(t)$  are quotients of  $\mathcal{O}_{\mathbb{P}^n}(-m)^{p(m)}$ , hence they form a bounded family.

One can almost get a uniform vanishing theorem for  $H^0(\mathbb{P}^n, F(-r))$  as follows. Let  $F$  be a coherent sheaf on  $\mathbb{P}^n$  and  $s \in H^0(\mathbb{P}^n, F)$  a section whose support has dimension  $d$ . Then

$$h^0(\mathbb{P}^n, F(m)) \geq h^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(m)) = \binom{m+d}{d}.$$

In particular, if  $F$  is  $m$ -regular and  $h^0(\mathbb{P}^n, F(m)) = \chi(\mathbb{P}^n, F(m)) =: r$  then every section of  $F(m-r)$  has 0-dimensional support.

**Lemma 32.** *Let  $G$  be a coherent sheaf on  $\mathbb{P}^n$  with Hilbert polynomial  $p(t)$ . Assume that  $G$  has no associated primes of dimension  $< 2$ . Let  $H \subset \mathbb{P}$  be a hyperplane that does not contain any of the associated primes of  $G$  and assume that  $G|_H$  is  $m_1$ -regular.*

*Then  $G$  is  $m$  regular for some  $m$  depending only on  $m_1$  and  $p(t)$ .*

Proof. Using the cohomology sequence of

$$0 \rightarrow G(r-1) \rightarrow G(r) \rightarrow G|_H(r) \rightarrow 0$$

we conclude that  $H^i(X, G(r-1)) \cong H^i(X, G(r))$  for  $i \geq 2$  and  $r \geq m_1 - i + 1$ . Thus, by Serre's vanishing,  $H^i(X, G(r)) = 0$  for  $i \geq 2$  and  $r \geq m_1 - i$ .

For  $i = 1$  we have only an exact sequence

$$H^0(X, G(r)) \xrightarrow{b(r)} H^0(X \cap H, G|_H(r)) \rightarrow H^1(X, G(r-1)) \xrightarrow{c(r)} H^1(X, G(r)) \rightarrow 0,$$

which shows that  $b(r)$  is onto iff  $c(r)$  is an isomorphism.

We also have a commutative square

$$\begin{array}{ccc} H^0(X, G(r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_X(1)) & \rightarrow & H^0(X, G(r+1)) \\ b(r) \otimes \sigma \downarrow & & \downarrow b(r+1) \\ H^0(X \cap H, G|_H(r)) \otimes H^0(H, \mathcal{O}_H(1)) & \xrightarrow{t(r)} & H^0(X \cap H, G|_H(r+1)) \end{array}$$

where  $\sigma : H^0(\mathbb{P}^n, \mathcal{O}_X(1)) \rightarrow H^0(H, \mathcal{O}_H(1))$  is the (surjective) restriction. Since  $G|_H$  is  $m_1$ -regular,  $t(r)$  is onto for  $r \geq m_1$  (31).

This shows that if  $b(r)$  is onto for some  $r \geq m_1$  then  $b(r+1)$  is also onto. Thus, if  $b(r)$  is onto then  $b(s)$  is onto for every  $s \geq r$  and  $c(s)$  is an isomorphism for every  $s \geq r$ . Again by Serre's vanishing, this gives that  $H^1(X, G(r)) = 0$ .

Otherwise  $H^1(X, G(r)) \neq 0$  but then  $h^1(X, G(r)) > h^1(X, G(r+1))$ . In either case we get that

$$H^1(X, G(r)) = 0 \quad \text{for } r \geq m_1 + h^1(X, G(m_1)).$$

Since  $h^1(X, G(m_1)) = h^0(X, G(m_1)) - p(m_1)$ , we are done if we can bound  $h^0(X, G(m_1))$  from above. Since  $G$  has no 0-dimensional associated primes,

$$h^0(X, G(m_1)) \leq \sum_{i \geq 0} h^0(H, G|_H(m_1 - i)),$$

and the latter sum is finite by (31) and bounded by induction if  $G|_H$  has no 0-dimensional associated primes. The latter follows from our assumptions.  $\square$

The following is proved in [Gro67, III.7.7.8–9], see also [LMB00, 4.6.2.1] and [Lie06, 2.1.3].

**Definition–Lemma 33.** Let  $f : X \rightarrow S$  be proper. Let  $F, L$  be coherent sheaves on  $X$  such that  $L$  is flat over  $S$ . Then there is an  $S$ -scheme  $\underline{\text{Hom}}(F, L)$  parametrizing homomorphisms from  $F$  to  $L$ . That is, for any  $g : T \rightarrow S$ , there is a natural isomorphism

$$\text{Hom}_T(g_X^* F, g_X^* L) \cong \text{Mor}_S(T, \underline{\text{Hom}}(F, L)),$$

where  $g_X : T \times_S X \rightarrow X$  is the fiber product of  $g$  with the identity of  $X$ .

Proof. Note that there is a natural identification between

- (1) homomorphisms  $\phi : F \rightarrow L$ , and
- (2) quotients  $\Phi : (F + L) \rightarrow M$  which induce an isomorphism  $\Phi|_L : L \cong M$ .

Let  $\pi : \text{Quot}(F + L) \rightarrow S$  denote the quot-scheme parametrizing quotients of  $F + L$  with universal quotient  $u : \pi_X^*(F + L) \rightarrow M$ , where  $\pi_X$  denotes the induced map  $\pi_X : \text{Quot}(F + L) \times_S X \rightarrow X$ .

Consider now the restriction of  $u$  to  $u_L : \pi_X^* L \rightarrow M$ . By the Nakayama lemma, for a map between sheaves it is an open condition to be surjective. For a surjective map with flat target, it is an open condition to be fiber-wise injective (cf. [Mat86, 22.5]). Thus there is an open subset

$$\text{Quot}^0(F + L) \subset \text{Quot}(F + L)$$

which parametrizes those quotients  $v : F + L \rightarrow M$  which induce an isomorphism  $v_L : L \cong M$ . Thus  $\underline{\text{Hom}}(F, L) = \text{Quot}^0(F + L)$ .  $\square$

### Push forward and $S_2$ .

Here we collect some well known results about normalization, pushing forward and  $S_2$ -sheaves.

**Lemma 34.** *Let  $R$  be a Noetherian ring and  $M$  an  $R$ -module. Assume that each associated prime of  $M$  has dimension  $n$ . The following are equivalent:*

- (1) *If  $r \in R$  is not contained in any associated prime of  $M$  then every associated prime of  $M/rM$  has dimension  $(n - 1)$ .*
- (2) *If  $N \supset M$  has the same associated primes as  $M$  and every associated prime of  $N/M$  has dimension  $\leq (n - 2)$  then  $N = M$ .*

Proof. To see (2)  $\Rightarrow$  (1), pick  $n \in N$  such that the associated primes of  $Rn/Rn \cap M$  have dimension  $\leq (n - 2)$ . There is an  $r \in R$  which is not contained in any associated prime of  $Rn \cap M$  such that  $rn \in M$ . Then  $rn$  leads to an associated prime of  $M/rM$  of dimension  $\leq (n - 2)$ . By (1) we obtain that  $rn \in rM$  and so  $n \in M$ .

Conversely, assume that there is a submodule  $rM \subset M' \subset M$  such that every associated prime of  $M'/rM$  has dimension  $\leq (n - 2)$ . Then  $N := r^{-1}M' \supset M$  shows that  $r^{-1}M' = M$ , Thus  $M' = rM$ , which gives (1)  $\Rightarrow$  (2).  $\square$

The proof of the next lemma is essentially the same as the coherence argument in (16).

**Lemma 35.** [Gro67, IV.5.11.1] *Let  $X$  be the spectrum of a Nagata ring  $R$ ,  $i : U \rightarrow X$  the immersion of an open set and  $F$  a coherent sheaf on  $U$ . Then  $i_*F$  is coherent iff  $\text{codim}(\bar{x} \cap (X \setminus U), \bar{x}) \geq 2$  for every associated prime  $x \in X$  of  $F$ .*

**Lemma 36.** *Let  $X$  be an affine scheme,  $i : U \rightarrow X$  an open immersion and  $W = X \setminus U$ . Let  $F$  be a coherent sheaf on  $U$  and  $G$  a quasi coherent sheaf on  $X$  such that  $G|_U \cong F$ . The following conditions are equivalent:*

- (1)  $G \cong i_*F$ .
- (2) For  $a, b \in I_W$ , if the sequence

$$0 \rightarrow F \xrightarrow{(b, -a)} F + F \xrightarrow{(a, b)} F$$

is exact then so is

$$0 \rightarrow G \xrightarrow{(b, -a)} G + G \xrightarrow{(a, b)} G.$$

- (3)  $\text{depth}_W(G) \geq 2$ .

- (4) If  $a \in I_W$  does not vanish on any associated prime of  $F$  then  $G/aG$  has no associated prime supported on  $W$ .
- (5)  $H_W^i(X, G) = 0$  for  $i = 0, 1$ .
- (6)  $H_W^0(X, G) = 0$  and for every coherent sheaf  $Q$  such that  $\text{Supp } Q \subset W$  every extension

$$0 \rightarrow G \rightarrow G' \rightarrow Q \rightarrow 0 \quad \text{splits.}$$

Proof. We first prove that (36.1) and (36.2) are equivalent. Since  $F$  is coherent, it has only finitely many associated primes. Choose  $a \in I_W$  which is not contained in any associated prime of  $F$  and  $b \in I_W$  which is not contained in any associated prime of  $F/aF$ .

Then  $0 \rightarrow F \xrightarrow{(b, -a)} F + F \xrightarrow{(a, b)} F$  is exact. Since  $i_*$  is left exact, this implies (36.2). Conversely, assume (36.2).  $G|_U \cong F$ , thus there is a natural homomorphism  $G \rightarrow i_*F$ . Let  $K \subset G$  be its kernel. Every element of  $K$  is killed by a power of  $I_W$ , thus if  $K \neq 0$  then we can choose  $0 \neq g \in K$  such that  $I_W \cdot g = 0$ , in particular  $ag = bg = 0$ . This is impossible since (36.2) is left exact.

Thus  $G \rightarrow i_*F$  is an injection; let  $C$  be its cokernel. Every element of  $C$  is killed by a power of  $I_W$ , thus if  $C \neq 0$  then we can choose  $g' \in i_*F \setminus G$  such that  $ag', bg' \in G$ .  $(bg', -ag') \in G + G$  is in the kernel of the map  $(a, b)$ , hence by exactness there is a  $g \in G$  such that  $bg = bg'$  and  $ag = ag'$ . This implies that  $g - g' = 0$ , a contradiction.

(36.2) implies that  $(a, b)$  is a  $G$ -sequence of length two, hence  $\text{depth}_W(G) \geq 2$ . Conversely, if  $\text{depth}_W(G) \geq 1$  then none of the associated primes of  $G$  are contained in  $W$ . Every other associated prime of  $G$  is also an associated prime of  $F$ , hence  $G$  has only finitely many associated primes. Therefore one can choose  $a \in I_W$  which is not contained in any associated prime of  $F$ . If  $\text{depth}_W(G) \geq 2$  then none of the associated primes of  $G/aG$  are contained in  $W$ , hence one can choose  $b \in I_W$  which is not contained in any associated prime of  $G/aG$ . This shows that (36.2) is equivalent to (36.3).

(36.4) is a restatement of (36.3).

For any quasi coherent sheaf  $G$  there is an exact sequence

$$0 \rightarrow H_W^0(X, G) \rightarrow H^0(X, G) \rightarrow H^0(U, G|_U) \rightarrow H_W^1(X, G) \rightarrow H^1(X, G).$$

Thus  $H_W^0(X, G) = 0$  iff  $G \rightarrow i_*(G|_U)$  is an injection. Since  $X$  is affine,  $H^1(X, G) = 0$  thus  $H_W^1(X, G) = 0$  iff  $G \rightarrow i_*(G|_U)$  is a surjection. These show that (36.1) and (36.5) are equivalent.

If  $H_W^0(X, G) = 0$  then  $G \rightarrow i_*F$  is an injection. Thus if  $G \neq i_*F$  then we have a nonsplit extension. If  $G \rightarrow G'$  is as in (36.6) then  $G'|_U \cong F$  gives a homomorphism  $G' \rightarrow i_*F$  which is a splitting of  $G \rightarrow G'$  if  $G \cong i_*F$ .  $\square$

### Appendix: Algebraic space case. (joint with M. Lieblich)

We consider the case when  $S$  is a Noetherian algebraic space and  $f : X \rightarrow S$  is a proper morphism of algebraic spaces.

**37** (Flat families of coherent sheaves). Let  $f : X \rightarrow S$  be a proper morphism. The functor of flat families of coherent sheaves  $\text{Flat}(X/S)$  is represented by an algebraic stack  $\text{Flat}(X/S)$  which is locally of finite type but very nonseparated; cf. [LMB00, 4.6.2.1].



However, as in (10.1),  $\text{Flat}(X/S)$  satisfies the existence part of the valuative criterion of properness. That is, if  $T$  is the spectrum of a DVR with generic point  $t_g$  then every morphism  $t_g \rightarrow \text{Flat}(X/S)$  extends to  $T \rightarrow \text{Flat}(X/S)$ .

In fact, an even stronger property holds:

**37.1 Theorem.** [RG71, 5.2.2, 5.7.9] Let  $U/S$  be a separated  $S$ -space of finite type and  $g : U \rightarrow \text{Flat}(X/S)$  a morphism. Then there is an  $S$ -space  $\bar{U} \supset U$  which is proper over  $S$  such that  $g$  extends to a morphism  $\bar{g} : \bar{U} \rightarrow \text{Flat}(X/S)$ .

**38** (Construction of  $\text{QHusk}(F)$ ). Let  $\sigma : \text{Flat}(X/S) \rightarrow S$  be the structure morphism and let  $U_{X/S}$  denote the universal family over  $\text{Flat}(X/S)$ . There is an open substack

$$\text{Flat}^n(X/S) \subset \text{Flat}(X/S)$$

parametrizing those sheaves that are pure of dimension  $n$ . Let  $U_{X/S}^n$  be the corresponding universal family.

Consider  $X \times_S \text{Flat}^n(X/S)$  with coordinate projections  $\pi_1, \pi_2$ . The stack

$$\underline{\text{Hom}}(\pi_1^* F, \pi_2^* U_{X/S}^n)$$

parametrizes all maps from the sheaves  $F_s$  to sheaves  $N_s$  that are pure of dimension  $n$  (33).

We claim that  $\text{QHusk}(F)$  is an open substack of  $\underline{\text{Hom}}(\pi_1^* F, \pi_2^* U_{X/S}^n)$ . Indeed, as in the proof of (33), for a map of sheaves  $M \rightarrow N$  with  $N$  flat over  $S$ , it is an open condition to be surjective at the generic points of the support.

As in (10.1), we see that  $\text{QHusk}(F)$  is separated.

Putting these together, and using that an algebraic stack whose diagonal is a monomorphism is an algebraic space (see, for instance, [LMB00, Sec.8]), we obtain the first existence theorem:

**Theorem 39.** *Let  $f : X \rightarrow S$  be a proper morphism of algebraic spaces and  $F$  a coherent sheaf on  $X$ . Then*

- (1)  *$\text{QHusk}(F)$  is separated and it has a fine moduli space  $\text{QHusk}(F)$ .*
- (2) *Every irreducible component of  $\text{QHusk}(F)$  is proper over  $S$ .* □

**40** (Construction of  $\text{Hull}(F)$ ). In a flat family of coherent sheaves, it is an open condition to be  $S_2$  over their support and for a map to a flat sheaf it is also an open condition to be an isomorphism at the codimension 1 points of their support. This implies that  $\text{Hull}(F)$  is an open subspace of  $\text{QHusk}(F)$ .

We claim that  $\text{Hull}(F)$  is of finite type. First, it is locally of finite type since  $\text{QHusk}(F)$  is. Second, we claim that  $\text{red Hull}(F)$  is dominated by an algebraic space of finite type. In order to see this, consider the (reduced) structure map  $\text{red Hull}(F) \rightarrow \text{red } S$ . It is an isomorphism at the generic points, hence there is an open dense  $S^0 \subset \text{red } S$  such that  $S^0$  is isomorphic to an open subspace of  $\text{red Hull}(F)$ . Repeating this for  $\text{red } S \setminus S^0$ , by Noetherian induction we eventually write  $\text{red Hull}(F)$  as a disjoint union of finitely many locally closed subspaces of  $\text{red } S$ . (We do not claim, however, that every irreducible component of  $\text{red Hull}(F)$  is a locally closed subspace of  $\text{red } S$ .)

These together imply that  $\text{Hull}(F)$  is of finite type. (Indeed, if  $U \rightarrow V$  is a surjection,  $U$  is of finite type and  $V$  is locally of finite type then  $V$  is of finite type.)

As in (18.2), the structure map  $\text{Hull}(F) \rightarrow S$  is a monomorphism. However, in the nonprojective case, it need not be a locally closed decomposition (though we do

not know any examples). We can summarize these considerations in the following theorem.

**Theorem 41** (Flattening decomposition for hulls). *Let  $f : X \rightarrow S$  be a proper morphism of algebraic spaces and  $F$  a coherent sheaf on  $X$ . Then*

- (1)  $\text{Hull}(F)$  is separated and it has a fine moduli space  $\text{Hull}(F)$ .
- (2)  $\text{Hull}(F)$  is an algebraic space of finite type over  $S$ .
- (3) The structure map  $\text{Hull}(F) \rightarrow S$  is a surjective monomorphism.  $\square$

In some cases one can see that  $\text{Hull}(F) \rightarrow S$  is a locally closed decomposition using the following valuative criterion of locally closed embeddings.

**Proposition 42.** *Let  $f : X \rightarrow Y$  be a morphism of finite type. Then  $f$  is a locally closed embedding iff*

- (1)  $f$  is a monomorphism, and
- (2) if  $T$  is the spectrum of a DVR and  $g : T \rightarrow Y$  a morphism such that  $g(T) \subset f(X)$  then  $g$  lifts to  $g_X : T \rightarrow X$ .

Proof. Since  $f$  is a monomorphism, it is quasi-finite. Take any proper  $\tilde{f} : \tilde{X} \rightarrow Y$  extending  $f$  and then its Stein factorization. We obtain an algebraic space  $\bar{X} \subset X$  and a finite morphism  $\bar{f} : \bar{X} \rightarrow Y$  extending  $f$ . Set  $Z := \bar{X} \setminus X$ . If  $Z = \bar{f}^{-1}\bar{f}(Z)$  then

$$f(X) = \bar{f}(\bar{X}) \setminus \bar{f}(Z),$$

and  $\bar{f}(Z) \subset \bar{f}(\bar{X}) \subset Y$  are closed embeddings. Thus  $f(X) \subset Y$  is locally closed and  $f : X \rightarrow f(X)$  is a proper monomorphism hence an isomorphism by (22).

Otherwise, there are points  $z \in Z$  and  $x \in X$  such that  $\bar{f}(z) = \bar{f}(x)$ . Let  $T$  be the spectrum of a DVR and  $h : T \rightarrow \bar{X}$  a morphism which maps the closed point to  $z$  and the generic point to  $X$ . Set  $g := \bar{f} \circ h$ . Then  $g(T) \subset f(X)$  and the only lifting of  $g$  to  $T \rightarrow X$  is  $h$ , but  $h(T) \not\subset X$ .  $\square$

**43** (Proof of (21)). One can get another proof of (21) using (41) and (42) as follows.

Since  $f : X \rightarrow S$  is projective, there is an  $f$ -ample divisor  $H$  and we can decompose  $\text{Hull}(F) = \coprod_p \text{Hull}_p(F)$  according to the Hilbert polynomials. In order to prove that each  $\text{Hull}_p(F) \rightarrow S$  is a locally closed embedding, we check the valuative criterion (42).

We have  $X_T \rightarrow T$  and a coherent sheaf  $F_T$  such that the hulls of  $F_{t_g}$  and of  $F_{t_0}$  have the same Hilbert polynomials.

Let  $F_{t_g} \rightarrow G_{t_g}$  be the hull over the generic point and extend  $G_{t_g}$  to a husk  $F_T \rightarrow G_T$ .

Let  $G_{t_0}^{[**]}$  denote the hull of  $G_{t_0}$ . Then the composite  $F_{t_0} \rightarrow G_{t_0} \rightarrow G_{t_0}^{[**]}$  is the hull of  $F_{t_0}$ . By assumption and by flatness

$$\chi(X_0, G_{t_0}^{[**]}(tH)) = p(t) = \chi(X_0, G_{t_g}(tH)) = \chi(X_0, G_{t_0}(tH)).$$

Hence, by (26.2),  $G_{t_0} = G_{t_0}^{[**]}$  and so  $G_{t_0} = G_{t_0}^{[**]}$  is the hull of  $F_{t_0}$ . Thus  $G_T$  defines the lifting  $T \rightarrow \text{Hull}_p(F)$ .  $\square$

**Acknowledgments .** I thank D. Abramovich, J. Lipman and D. Rydh for many useful comments and corrections. Partial financial support was provided by the NSF under grant number DMS-0500198.

## REFERENCES

- [AH09] Dan Abramovich and Brendan Hassett, *Stable varieties with a twist*, <http://www.citebase.org/abstract?id=oai:arXiv.org:0904.2797>, 2009.
- [AK06] Valery Alexeev and Allen Knutson, *Complete moduli spaces of branchvarieties*, <http://www.citebase.org/abstract?id=oai:arXiv.org:math/0602626>, 2006.
- [Gro67] Alexander Grothendieck, *Éléments de géométrie algébrique. I–IV.*, Inst. Hautes Études Sci. Publ. Math. (1960–67), no. 4,8,11,17,20,24,28,32.
- [Gro62] ———, *Techniques de construction et théorèmes d’existence en géométrie algébrique. IV. Les schémas de Hilbert*, Séminaire Bourbaki, Vol. 6, Soc. Math. France, Paris, 1962, pp. 249–276, Exp. No. 221. MR MR1611822
- [Hac04] Paul Hacking, *Compact moduli of plane curves*, Duke Math. J. **124** (2004), no. 2, 213–257. MR MR2078368 (2005f:14056)
- [Hon04] M. Honsen, *A compact moduli space parametrizing Cohen-Macaulay curves in projective space*, Ph.D. Thesis, MIT, 2004.
- [KM97] Seán Keel and Shigefumi Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), no. 1, 193–213. MR MR1432041 (97m:14014)
- [Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 32, Springer-Verlag, Berlin, 1996. MR MR1440180 (98c:14001)
- [Kol97] ———, *Quotient spaces modulo algebraic groups*, Ann. of Math. (2) **145** (1997), no. 1, 33–79. MR MR1432036 (97m:14013)
- [Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. I–II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 48–49, Springer-Verlag, Berlin, 2004. MR MR2095471 (2005k:14001a)
- [Lie06] Max Lieblich, *Remarks on the stack of coherent algebras*, Int. Math. Res. Not. (2006), Art. ID 75273, 12. MR MR2233719 (2008c:14022)
- [LMB00] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 39, Springer-Verlag, Berlin, 2000. MR MR1771927 (2001f:14006)
- [LP93] Joseph Le Potier, *Systèmes cohérents et structures de niveau*, Astérisque (1993), no. 214, 143. MR MR1244404 (95e:14005)
- [LP95] ———, *Faisceaux semi-stables et systèmes cohérents*, Vector bundles in algebraic geometry (Durham, 1993), London Math. Soc. Lecture Note Ser., vol. 208, Cambridge Univ. Press, Cambridge, 1995, pp. 179–239. MR MR1338417 (96h:14010)
- [Mat86] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986, Translated from the Japanese by M. Reid. MR MR879273 (88h:13001)
- [Mum66] David Mumford, *Lectures on curves on an algebraic surface*, With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59, Princeton University Press, Princeton, N.J., 1966. MR MR0209285 (35 #187)
- [PT07] R. Pandharipande and R. P. Thomas, *Curve counting via stable pairs in the derived category*, <http://www.citebase.org/abstract?id=oai:arXiv.org:0707.2348>, 2007.
- [RG71] Michel Raynaud and Laurent Gruson, *Critères de platitude et de projectivité. Techniques de “platification” d’un module*, Invent. Math. **13** (1971), 1–89. MR MR0308104 (46 #7219)
- [Ryd08] David Rydh, *Families of cycles and the Chow scheme*, Ph.D. Thesis, KTH, Stockholm, 2008.
- [Ser06] Edoardo Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften, vol. 334, Springer-Verlag, Berlin, 2006. MR MR2247603 (2008e:14011)

Princeton University, Princeton NJ 08544-1000  
 kollar@math.princeton.edu